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A MINIMUM PRINCIPLE FOR TRACTIONS IN THE ELASTOSTATICS OF CABLE NETWORKS†

MARIO CANNAROZZI
Istituto di Scienza delle Costruzioni, Università di Bologna, I-40136 Bologna, Italy

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Abstract—A constrained extremum principle for the elastostatics of cable networks is formulated. A convex, non-differentiable functional involving only static variables is shown to attain its minimum on a convex set, in correspondence of the solution of the problem. Taking into account slackening of cables, existence and uniqueness are proved for the solution. Finite element models can be developed on the grounds of the theory, as shown in some examples.

1. INTRODUCTION

Displacement approaches to problems of cable networks are very popular in the literature. Algebraic formulations have been developed with reference to problems of networks submitted to lumped loads (suspension roofs)[1–7]. Analytical formulations have been given to account for distributed loads[8–13], and solving dynamic problems[14–16]. Incremental methods of elastic, as well as elastoplastic analysis have been proposed, but the stress-unilateral behaviour of the cables has been taken into account explicitly only sporadically (see, e.g. Ref. [17]).

As a matter of fact, statics of cables and cable networks are characterized both by geometrical (large displacements) and mechanical (stress-unilateral behaviour) nonlinearities. In this context, the author has proposed a constrained, stationary formulation for the statics of elastic networks submitted to conservative (but generic) loads, from which two complementary, constrained extremum formulations have been shown to stem, corresponding to the usual principles of the total minimum potential energy and complementary energy, this one being expressed in terms of both static and geometric variables[18].

In this paper, the latter formulation is reconsidered (Section 2), and a new constrained minimum problem for a convex functional in the static variables only is deduced. Existence and uniqueness for the solution of this problem on the convex set of the admissible (i.e. tensile) tractions are proved (Section 3). The functional is recognized to be non-Gateaux differentiable at the point where it attains its minimum, if slackening of cables occurs. The relevant minimum conditions, taking into account slackening, are then deduced. Such conditions coincide with the ones achievable via the usual variational procedure if the cables are all in tension at the equilibrium configuration, and they are the compatibility relationships for the network, in the framework of the assumed formulation (Section 4).

The theory is available for developing finite element equilibrium approaches, which seem particularly convenient in the analysis of cable systems, as shown in some simple applications presented at the end of the paper.

2. GENERAL REMARKS

The cable is considered as a unidimensional solid in $\mathbb{R}^3$. Any configuration C of the cable is described by the coordinates of its points in an orthogonal Cartesian reference frame $(0, x_i;\ i = 1, 2, 3)$. A reference configuration C^0 is assumed for the cable, where s is the curvilinear abscissa along C^0. The cable is unstrained in this configuration, and its

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length is L. The points of C are related to C^0 by assuming that their coordinates are expressed through s, i.e. by letting $x_i = x_i(s)$, $s \in (0, L)$. The end points of the cable have coordinates $X_i^0 = x_i(0)$, $X_i^L = x_i(L)$.

The strain at a point on the cable is defined in terms of relative elongation by†

$$\eta = \frac{d\bar{s} - ds}{ds} = (x_{i/s} x_{i/s})^{1/2} - 1 \quad (1)$$

where $d\bar{s}$ is the length of the (infinitesimal) cable element in C.

The cable is submitted to a distributed, conservative load, whose components $q_i(s)$ are forces per unit of undeformed length. The forces F_i^0 and F_i^L act at the ends of the cable, $s = 0$ and L, respectively. The traction in the Lagrangean sense (i.e. referred to the dimension of the unstrained cable element) along the cable is denoted by $T(s)$. Traction T is related to the actual traction T^* by $T = T^*/(1 + \eta)$, and T is assumed positive if the cable is stretched.

Equilibrium is expressed by the equations

$$(T x_{i/s})_{/s} + q_i = 0, \qquad s \in (0, L) \quad (2)$$

$$(T x_{i/s})_0 = -F_i^0, \qquad s = 0; \qquad (T x_{i/s})_L = F_i^L, \qquad s = L. \quad (3)$$

If the coordinates of any of the cable ends are prescribed, i.e. this end is restrained, eqns (3) define the relevant restraint forces.

The strain conjugated with traction T is defined by

$$\lambda = \frac{1}{2}(x_{i/s} x_{i/s} - 1) \quad (4)$$

and strains η and λ are related as follows

$$\eta = (1 + 2\lambda)^{1/2} - 1. \quad (5)$$

As the cable can bear only tensile tractions, the elastic part of strain λ, denoted by ε, must be nonnegative. Taking into account def. (4), this unilateral behaviour is described by the compatibility relationships

$$\bar{g} = \varepsilon - \frac{1}{2}(x_{i/s} x_{i/s} - 1) + \bar{\varepsilon} \geqslant 0, \qquad \varepsilon \geqslant 0, \qquad \langle \varepsilon, \bar{g} \rangle = 0, \qquad s \in (0, L) \quad (6)$$

where $\bar{\varepsilon}$ denotes a possible initial strain ($\bar{\varepsilon} > -1/2$).

Strain ε and traction T are related by the elastic constitutive law

$$\varepsilon = \frac{dG(T)}{dT}, \qquad \varepsilon(0) = 0 \quad (7a, b)$$

where the strain energy density $G(T)$ is admitted to be a continuous, strictly convex function defined for $T \geqslant 0$. This implies

$$\varepsilon(T) > 0, \qquad \forall T > 0. \quad (7c)$$

† The derivative of a (differentiable) function $f(s)$ with respect to s is denoted by $f_{/s}$. Repeated subscripts mean summation over the range 1, 2, 3, $\langle \cdot, \cdot \rangle$ means scalar product between two square integrable functions.

Moreover, the following assumption is stipulated

$$\frac{d\varepsilon}{dT} \geqslant \alpha, \qquad \alpha > 0, \qquad \forall T \geqslant 0. \tag{7d}$$

Taking into account the constitutive law, relations (6) are transformed into the elastokinematic relationships[18]

$$g = \frac{dG}{dT} - \frac{1}{2}(x_{i/s}x_{i/s} - 1) + \bar{\varepsilon} \geqslant 0, \quad T \geqslant 0, \quad \langle T, g \rangle = 0, \quad s \in (0, L). \tag{8}$$

Consider now a network of n cables and l free nodes. The set of such nodes is denoted by K. The unstrained length of the jth cable, $j = 1, 2, \ldots, n$, is denoted by L^j, and the (unknown) coordinates of the kth node, $k = 1, 2, \ldots, l$, are denoted by X_i^k, $i = 1, 2, 3$. The network is restrained at further m nodes, whose (prescribed) coordinates are denoted by $\bar{X}_i^h$, $h = 1, 2, \ldots, m$, $i = 1, 2, 3$. The set of such nodes is denoted by H. A distributed load of components q_i^j acts on cable j, and a concentrated load of components Q_i^k acts on node k. If the jth cable is connected to a free node $k \in K$ by its initial, $s = 0$ (terminal, $s = L^j$), end, index j is entered into the set I^k (E^k). If the same node is restrained, index j of the cable is entered into the set I^h (E^h).

Equilibrium and compatibility (i.e. continuity) conditions at the kth node read

$$\sum_{j \in I^k} (F_i^0)^j + \sum_{j \in E^k} (F_i^L)^j = Q_i^k, \qquad k = 1, 2, \ldots, l \tag{9}$$

$$x_i^j(0) = X_i^k, \qquad j \in I^k; \qquad x_i^j(L^j) = X_i^k, \qquad j \in E^k. \tag{10}$$

The end forces on a cable restrained at node h are given by eqns (3), while the coordinates of the end points

$$x_i^j(0) = \bar{X}_i^h, \qquad j \in I^h; \qquad x_i^j(L^j) = \bar{X}_i^h, \qquad j \in E^h \tag{11}$$

should be prescribed.

Relationships (2), (3), (8)–(11) rule the problem of the equilibrium of an elastic network. A variational formulation of the same problem lies on the stationary (minimum) of the functional[18]†

$$\Phi(T, x_i) = \sum_j \left\{ \int_0^L \left[G + \frac{1}{2} T(x_{i/s}x_{i/s} + 1) + T\bar{\varepsilon} \right] ds \right\} - \sum_h \left\{ \bar{X}_i^h \left[\sum_{j \in E^h} (Tx_{i/s})_L - \sum_{j \in I^h} (Tx_{i/s})_0 \right] \right\} \tag{12}$$

on the set of solutions of the equilibrium equations (2), (3), (9), which fulfil the admissibility condition $T \geqslant 0$ on tractions. If the variables F_i^0 and F_i^L are eliminated between eqns (3) and (9), the above problem takes the form

$$\min \Phi(T, x_i) \tag{13a}$$

† Index j denoting functions or parameters for the jth cable will be omitted from now on, unless a misunderstanding is possible.

subjected to

$$[(Tx_{i/s})_{/s} + q_i]^j = 0, \qquad j = 1, 2, \dots, n \tag{13b}$$

$$\sum_{j\in I^k} (Tx_{i/s})_0 - \sum_{j\in E^k} (Tx_{i/s})_L + Q_i^k = 0, \qquad k = 1, 2, \dots, l \tag{13c}$$

$$T^j \geqslant 0, \qquad j = 1, 2, \dots, n. \tag{13d}$$

3. A PRINCIPLE OF MINIMUM FOR TRACTIONS

An extremum formulation in the static variables alone can be obtained from problem (13) above.

Equations (13b) yield for the jth cable

$$Tx_{i/s} = -[R_i(s) + F_i^0], \qquad R_i(s) = \int_0^s q_i(\tau)\,\mathrm{d}\tau \tag{14a, b}$$

then

$$(Tx_{i/s})_0 = -F_i^0, \qquad (Tx_{i/s})_L = -[R_i(L) + F_i^0]. \tag{14c, d}$$

Equation (14c) defines F_i^0 as the ith component of the force acting at the initial end of the cable. Assume, as natural, that load q_i is represented by a bounded, generally continuous function of the abscissa s. Then, $R_i(s)$ is a bounded, absolutely continuous function on the interval $(0, L)$.

For each $T \geqslant 0$, define the sets

$$w^+ = \{s | T(s) > 0\} \quad \text{and} \quad w^0 = \{s | T(s) = 0\}, \qquad w^+ \cup w^0 = (0, L). \tag{15a, b}$$

For any two given (T, F_i^0), eqn (14a) defines $x_{i/s}$ only if $s \in w^+$

$$x_{i/s} = -\frac{1}{T}[R_i(s) + F_i^0], \qquad s \in w^+. \tag{16}$$

On the other hand, any (no) finite value for $x_{i/s}$ can fulfil eqn (14a), if F_i^0 is such that $R_i(s) + F_i^0 = 0$ ($R_i(s) + F_i^0 \neq 0$), $s \in w^0$. Let, according to eqn (14a)

$$A(s) = \frac{1}{2} Tx_{i/s}x_{i/s} = -\frac{1}{2}[R_i(s) + F_i^0]x_{i/s}$$

and, for the above considerations, define

$$A(s) = \frac{1}{2T}[R_i(s) + F_i^0][R_i(s) + F_i^0], \qquad \forall s \in w^+$$

$$A(s) = 0, \qquad s \in w^0, \qquad s\!: [R_i(s) + F_i^0][R_i(s) + F_i^0] = 0$$

$$A(s) = \infty, \qquad s \in w^0, \qquad s\!: [R_i(s) + F_i^0][R_i(s) + F_i^0] \neq 0.$$

Then, taking into account eqns (14c) and (14d), functional Φ takes the form

$$\Phi_c(T, F_i^0) = \sum_j \left\{ \int_0^L [\varphi_1(T) + \varphi_2(T, F_i^0)]\,\mathrm{d}s \right\} + \mathscr{F}(F_i^0) \tag{17a}$$

where

$$\varphi_1(T) = G(T) + T\left(\frac{1}{2} + \bar{\varepsilon}\right) \tag{17b}$$

$$\varphi_2(T, F_i^0) = \frac{1}{2T}[R_i(s) + F_i^0][R_i(s) + F_i^0], \qquad \forall T > 0 \tag{17c}$$

$$\mathscr{F}(F_i^0) = -\sum_h \left\{ \bar{X}_i^h \left[\sum_{j\in I^h} (F_i^0) - \sum_{j\in E^h} (R_i(L) + F_i^0) \right] \right\} \tag{17d}$$

with the specifications

$$\varphi_2 = 0, \qquad \text{if } [R_i(s) + F_i^0][R_i(s) + F_i^0] = 0, \qquad \forall T \geqslant 0 \tag{17e}$$

$$\varphi_2 = \infty, \qquad \text{if } [R_i(s) + F_i^0][R_i(s) + F_i^0] \neq 0, \qquad T = 0 \tag{17f}$$

$$\Phi_c = \infty, \qquad \text{if } \exists j: \varphi_2 \notin L_1(0, L^j) \tag{17g}$$

and stipulating that $\varphi_2 \notin L_1(0, L^j)$, if def. (17f) is fulfilled on a set having positive measure.

It is worth noting that defs (17f) and (17g), taking into account (14b), cause functional Φ_c to assume infinite value for any $T(s)$ such that the traction is zero on a loaded, finite part of a cable ($q_i(s) \neq 0$, $s \in w^0$).

Taking into account eqns (14c) and (14d), the equilibrium conditions are rewritten as

$$\sum_{j\in I^k} (F_i^0) - \sum_{j\in E^k} (F_i^0) = Q_i^k + \sum_{j\in E^k} R_i(L), \qquad k = 1, 2, \ldots, l \tag{18}$$

and they become a system of $3l$ linear equations in the $3n$ variables $(F_i^0)^j$, $j = 1, 2, \ldots, n$, $n > l$. Let the network be restrained, i.e. one at least of its nodes is fixed. Then, any free node is connected with a/the restrained node directly or else through a sequence of cables. As a consequence, for any given set of nodal loads at least one subset of l cables, i.e. a subnetwork, can be found, which is able to carry the loads. In other words, equilibrium is possible for every system of nodal loads, hence the equilibrium equations are linearly independent, and the coefficient matrix of system (18) is a full rank matrix.

Problem (13) can be now rewritten in terms of static variables only

$$\min \Phi_c \tag{19a}$$

$$\sum_{j\in I^k} (F_i^0) - \sum_{j\in E^k} (F_i^0) = Q_i^k + \sum_{j\in E^k} R_i(L), \qquad k = 1, 2, \ldots, l \tag{19b}$$

$$T \geqslant 0 \tag{19c}$$

where Φ_c is defined by relationships (17).

Relationships (17) define the functional only for $T \geqslant 0$, as function G is defined only for non-negative values of T. However, this fact does not cause any loss of generality, as constraint (19c) rules out negativity for T.

Assume T^j as belonging to $L_2(0, L^j)$, and assume $(F_i^0)^j$ as a vector of $\mathbb{R}^3$. Then functional $\Phi_c(T, F_i^0)$ is defined[19] on the space

$$\mathscr{S} = \prod_{j=1}^{n} [L_2(0, L^j) \times \mathbb{R}^3], \qquad \|\cdot\| = \left\{ \sum_j \left[\int_0^L T^2 \, \mathrm{d}s + F_i^0 F_i^0 \right] \right\}^{1/2}$$

which is a Hilbert space, hence reflexive. Constraints (19c) define a convex, strongly closed set $\mathscr{W}$ of $\mathscr{S}$. Equations (19b) define a not empty linear manifold $\mathscr{L}$. Hence the set

$$\mathscr{D} \equiv \mathscr{W} \cap \mathscr{L}, \qquad \mathscr{D} \subset \mathscr{S}$$

i.e. the feasible set for problem (19), is convex and strongly closed. Moreover, for any $(T, F_i^0) \in \mathscr{D}$, Φ_c is strictly convex, coercive and lower semicontinuous (l.s.c.) (Appendix I). As functional Φ_c is convex, it is also weakly l.s.c. on $\mathscr{D}$[20, p. 11]. For any given real number b, define the (level) set

$$E_b = \{(T, F_i^0) \mid \Phi_c(T, F_i^0) \leqslant b, (T, F_i^0) \in \mathscr{D}\}.$$

As Φ_c is convex, coercive and l.s.c., and $\mathscr{D}$ is convex and closed, the set E_b is bounded, convex, closed, and therefore also weakly closed. Moreover, E_b is also weakly compact, because $\mathscr{S}$ is reflexive[21]. As a consequence, functional Φ_c is bounded from below and attains its minimum on the set E_b. As Φ_c is strictly convex, its point of minimum is unique[22].

4. THE MINIMUM CONDITIONS

It is worth noting that functional $\Phi_c(T, F_i^0)$, eqns (17), is Gateaux differentiable (G-diff) only in a subset of those points $P \equiv (T, F_i^0) \in \mathscr{D}$ where it takes finite values. In particular, Φ_c is not G-diff in the points where $T^j = 0$ on a set having positive measure for some j, because

$$\int_0^L \varphi_2 \, \mathrm{d}s$$

(defs (17c), (17e) and (17f)) is not G-diff if $T = 0$ on a set having positive measure. By this reason, it is impossible to derive the minimum conditions for Φ_c on $\mathscr{D}$ in conformity with the usual variational procedure.

A useful remark (Appendix II) should be premised at this point. Consider the functional

$$\psi(T, F_i^0) = \begin{cases} \int_w \varphi_2 \, \mathrm{d}s, & \text{if } \varphi_2 \in L_1(w) \\ \infty, & \text{otherwise} \end{cases}$$

$$(T, F_i^0) \in L_2(w) \times \mathbb{R}^3$$

which is defined on $\mathscr{P} \equiv T > 0$. Then, there exists

$$\delta\psi(\hat{P}; (\bar{P} - \hat{P})) = \lim_{\lambda \to 0} \frac{1}{\lambda} \{\psi(\hat{P} + \lambda(\bar{P} - \hat{P})) - \psi(\hat{P})\}, \qquad \hat{P}, \bar{P} \in \mathscr{P}$$

if

$$\left.\frac{\partial \varphi_2}{\partial T}\right|_{\hat{P}} = -\frac{1}{2\hat{T}^2}[R_i(s) + \hat{F}_i^0][R_i(s) + \hat{F}_i^0] \in L_2(w), \qquad \frac{1}{\hat{T}} \in L_1(w) \tag{20a, b}$$

and ψ is G-diff in $\hat{P}$ in the direction $(\bar{P} - \hat{P})$. If only eqn (20a) holds, ψ is G-diff in $\hat{P}$ only with respect to T.

Let $\hat{P} \equiv (\hat{T}, \hat{F}_i^0)$ be the point of $\mathscr{D}$ where Φ_c attains its minimum. Let U be the set collecting the indices, j, of the cables which are slackened at $\hat{P}$ in some part of finite extent, but not everywhere, as well as the indices of the cables whose traction tends to zero in the vicinity of a single point. Let $\bar{n} > 0$ be an integer such that the set

$$(w_{\bar{n}})^j = \left\{ s \mid \hat{T}^j(s) > \frac{1}{\bar{n}} \right\}, \qquad \forall j \in U$$

has positive measure. Then, for any $n \geq \bar{n}$ the sets

$$(w_n^+)^j = \left\{ s \mid \hat{T}^j(s) > \frac{1}{n} \right\} \quad \text{and} \quad (w_n^0)^j = \left\{ s \mid \hat{T}^j(s) \leq \frac{1}{n} \right\}$$

have positive measure, and are related as follows

$$(w_n^+)^j \cup (w_n^0)^j = (0, L^j), \qquad (w_n^+)^j \cap (w_n^0)^j = \varnothing, \qquad \forall n \geq \bar{n}.$$

The set $(w_n^+)^j$ tends to $(w^+)^j$, eqn (15a), for $n \to \infty$. The set $(w^+)^j$ coincides with the in tension parts of the jth cable, or with the whole cable if T^j is zero only in a set of single points. Moreover, the set $(w_n^0)^j$ tends to $(w^0)^j$, eqn (15b), which coincides with the slackened parts of the jth cable, and its measure is zero if the cable is (almost) everywhere in tension.

The above definitions can be extended to the cables where traction $\hat{T}$ is away from zero, or else zero over the whole length. The indices j of such cables are collected in the sets V and Z, respectively. $\hat{T}^j \geq \delta > 0$ a.e. holds in the first case ($j \in V$), and an index $\hat{n}$ can be found such that, for $n \geq \hat{n}$, the set $(w_n^+)^j$, $\forall j \in V$, coincides with the whole cable, while the set $(w_n^0)^j$ is of zero measure. $\hat{T}^j = 0$ a.e. holds in the second case ($j \in Z$), hence the set $(w_n^0)^j$ coincides with the whole cable for each n, while the set $(w_n^+)^j$ is of zero measure.

Consider a point $P \equiv (T, F_i^0) \in \mathscr{D}$. As $\mathscr{D}$ is a convex set, the point:

$$P^* = (1 - \lambda)\hat{P} + \lambda P = \hat{P} + \lambda(P - \hat{P}), \qquad \lambda \in (0, 1) \tag{21}$$

belongs to $\mathscr{D}$, and

$$\Phi_c(P^*) - \Phi_c(\hat{P}) \geq 0, \qquad \lambda \in (0, 1), \qquad \forall P \in \mathscr{D} \tag{22}$$

because Φ_c attains its minimum at point $\hat{P}$.

Let $n^* = \max\{\bar{n}, \hat{n}\}$, then relation (22) can be written, for each $n \geq n^*$, in the form

$$\begin{aligned} &\sum_j \left\{ \int_{w_n^+} [\varphi_1(P^*) - \varphi_1(\hat{P}) + \varphi_2(P^*) - \varphi_2(\hat{P})] \, \mathrm{d}s \right. \\ &\left. + \int_{w_n^0} [\varphi_1(P^*) - \varphi_1(\hat{P}) + \varphi_2(P^*) - \varphi_2(\hat{P})] \, \mathrm{d}s \right\} \\ &+ \mathscr{F}(P^*) - \mathscr{F}(\hat{P}) \geq 0, \qquad \lambda \in (0, 1), \qquad \forall P \in \mathscr{D}, \qquad \forall n \geq n^* \end{aligned}$$

which, as φ_2 is a convex function[23], implies

$$\begin{aligned} &\sum_j \left\{ \frac{1}{\lambda} \int_{w_n^+} [\varphi_1(P^*) - \varphi_1(\hat{P}) + \varphi_2(P^*) - \varphi_2(\hat{P})] \, \mathrm{d}s \right. \\ &+ \frac{1}{\lambda} \int_{w_n^0} [\varphi_1(P^*) - \varphi_1(\hat{P})] \, \mathrm{d}s \\ &\left. + \int_{w_n^0} [\varphi_2(P) - \varphi_2(\hat{P})] \, \mathrm{d}s \right\} + \mathscr{F}(P) - \mathscr{F}(\hat{P}) \geq 0, \\ &\lambda \in (0, 1), \qquad \forall P \in \mathscr{D}, \qquad \forall n \geq n^*. \end{aligned} \tag{23}$$

Since $\hat{T} > (1/n)$ on w_n^+, the functions $(\partial\varphi_2/\partial T)|_{\hat{P}}$ and $1/\hat{T}$ are bounded on w_n^+, hence they fulfil conditions (20). Then, by def. (21) of P^*, the limit for $\lambda \to 0$ of the left-hand side of inequality (23) can be taken, and the following inequality is obtained

$$\sum_j \left\{ \int_{w_n^+} \left[\left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}T}\bigg|_{\hat{P}} + \frac{\partial\varphi_2}{\partial T}\bigg|_{\hat{P}} \right)(T - \hat{T}) + \frac{\partial\varphi_2}{\partial F_i^0}\bigg|_{\hat{P}} (F_i^0 - \hat{F}_i^0) \right] \mathrm{d}s \right.$$

$$\left. + \int_{w_n^0} \left[\frac{\mathrm{d}\varphi_1}{\mathrm{d}T}\bigg|_{\hat{P}} (T - \hat{T}) + \varphi_2(P) - \varphi_2(\hat{P}) \right] \mathrm{d}s \right\}$$

$$+ \mathscr{F}(P) - \mathscr{F}(\hat{P}) \geqslant 0, \qquad \forall P \in \mathscr{D}, \qquad \forall n \geqslant n^*. \tag{24}$$

The above condition is necessary for the minimum of problem (19). Moreover, it is also sufficient. Indeed, if the left-hand side of inequality (24) is denoted by χ, the following inequality holds because of the convexity of φ_1 and φ_2

$$\chi \leqslant \sum_j \left\{ \int_{w_n^+} [\varphi_1(P) - \varphi_1(\hat{P}) + \varphi_2(P) - \varphi_2(\hat{P})]\,\mathrm{d}s \right.$$

$$\left. + \int_{w_n^0} [\varphi_1(P) - \varphi_1(\hat{P}) + \varphi_2(P) - \varphi_2(\hat{P})]\,\mathrm{d}s \right\} + \mathscr{F}(P) - \mathscr{F}(\hat{P}) \tag{25}$$

and its right-hand side is nonnegative if inequality (24) is satisfied, hence $\Phi_c(P) \geqslant \Phi_c(\hat{P})$, $\forall P \in \mathscr{D}$.

Observe that the integrals on domains w_n^0 and w_n^+, pertaining to the cables of index $j \in V$ and index $j \in Z$, respectively, do not appear in inequality (24). As a consequence, the minimum condition for problem (19) can be obtained via the usual variational procedure if all the cables are in tension ($j \in V$, $w_n^+ \equiv (0, L)$) at the solution of the problem.

A set of conditions is deduced from inequality (24), which has a transparent mechanical meaning. For any point $\bar{P} \in \mathscr{D}$ such that

$$\bar{T}^k = \hat{T}^k, \qquad k \neq j; \qquad \bar{T}^j(s) = \hat{T}^j(s), \qquad s \in (w_n^0)^j; \qquad \bar{F}_i^0 = \hat{F}_i^0$$

inequality (24) takes the form

$$\int_{(w_n^+)^j} \left\{ \frac{\mathrm{d}G}{\mathrm{d}T}\bigg|_{\hat{T}} - \frac{1}{2}\left[\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}^2} - 1 \right] + \bar{\varepsilon} \right\} (\bar{T} - \hat{T})\,\mathrm{d}s \geqslant 0$$

from which, for the arbitrariness of $\bar{T}^j$ on (w_n^+)

$$\frac{\mathrm{d}G}{\mathrm{d}T}\bigg|_{\hat{T}} - \frac{1}{2}\left[\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}^2} - 1 \right] + \bar{\varepsilon} = 0. \tag{26}$$

As the above condition is true for any $n \geqslant n^*$, it can be extended to the set $(w^+)^j$ defined by (15a). Taking into account (16), condition (26) is recognized as the elastokinematic condition, eqn (8), for $T^j > 0$.

It follows from condition (26) that the function $(\partial\varphi_2/\partial T)$, evaluated in $(\hat{T}, \hat{F}_i^0)^j$, is bounded on $(w^+)^j$ (see Appendix III), then it fulfils condition (20a). As a consequence, the set of the left-hand sides of condition (26) for the cables of indices $j \in U \cup V$, coincides with the gradient, with respect to variables T, of the functional Φ_c, if the domains of integration are restricted to the in tension parts only. Otherwise, the same set coincides with the gradient of Φ_c with respect to T if each cable is slackened at most in a set of single points, in correspondence of the solution of the problem.

A number of further conditions can be extracted from inequality (24), in addition to condition (26), which pertains the in tension parts of the cables only. On this purpose, it is worth-while to observe that the value of functions (dG/dT) and $[R_i(s) + F_i^0]$ in $(\hat{T}, \hat{F}_i^0)^j$ is a.e. zero on $(w^0)^j$. The former function is zero because it represents strain ε in terms of traction, the latter is zero because $\hat{P}$ is the solution of the problem, hence $\Phi_c(\hat{P}) < \infty$. Moreover, the function

$$\left.\frac{\partial \varphi_2}{\partial F_i^0}\right|_{(\hat{T},\hat{F}_i^0)^j} = \frac{1}{\hat{T}}[R_i(s) + \hat{F}_i^0], \qquad s \in (w^+)^j$$

is bounded and therefore integrable on $(w^+)^j$, as the value of $(\partial\varphi_2/\partial T)$ in $(\hat{T}, \hat{F}_i^0)^j$ is bounded on $(w^+)^j$. For any fixed $n \geqslant n^*$, the set

$$(\Delta w_n)^j = (w_n^0)^j \backslash (w^0)^j = (w^+)^j \backslash (w_n^+)^j$$

(of zero measure if $j \in V \cup Z$) is defined for the jth cable.

By virtue of condition (26), for any $P \in \mathscr{D}$ inequality (24) takes the form

$$\begin{aligned}
\sum_j \Bigg\{ & \int_{w^+} \frac{(R_i + \hat{F}_i^0)}{\hat{T}}(F_i^0 - \hat{F}_i^0)\,ds \\
& + \int_{w^0}\left[\frac{(F_i^0 - \hat{F}_i^0)(F_i^0 - \hat{F}_i^0)}{2T} + \left(\frac{1}{2} + \bar{\varepsilon}\right)T\right]ds \\
& + \int_{\Delta w_n}\left[\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{2\hat{T}^2}T + \frac{(R_i + F_i^0)(R_i + F_i^0)}{2T}\right. \\
& \left. - \frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}} - \frac{(R_i + \hat{F}_i^0)}{\hat{T}}(F_i^0 - \hat{F}_i^0)\right]ds\Bigg\} \\
& + \mathscr{F}(F_i^0) - \mathscr{F}(\hat{F}_i^0) \geqslant 0, \qquad \forall P \in \mathscr{D}, \qquad \forall n \geqslant n^*.
\end{aligned} \tag{27}$$

By minimizing the left-hand side of (27) with respect to T one obtains

$$T = \frac{[(F_i^0 - \hat{F}_i^0)(F_i^0 - \hat{F}_i^0)]^{1/2}}{1 + \bar{\eta}}, \qquad s \in w^0 \tag{28a}$$

$$T = [(R_i + F_i^0)(R_i + F_i^0)]^{1/2}\left\{\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}^2}\right\}^{-1/2}, \qquad s \in \Delta w_n \tag{28b}$$

where $\bar{\eta}$ denotes the strain η corresponding to $\bar{\varepsilon}$ (eqn (5)). Substitution of eqns (28) in (27) leads to

$$\begin{aligned}
\sum_j \Bigg\{ & (F_i^0 - \hat{F}_i^0)\int_{w^+} \frac{(R_i + \hat{F}_i^0)}{\hat{T}}\,ds + \Lambda\int_{w^0}(1 + \bar{\eta})\,ds \\
& + \int_{\Delta w_n} \frac{1}{\hat{T}}\{[(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)]^{1/2}[(R_i + F_i^0)(R_i + F_i^0)]^{1/2} \\
& -(R_i + \hat{F}_i^0)(R_i + F_i^0)\}\,ds\Bigg\} + \mathscr{F}(F_i^0) - \mathscr{F}(\hat{F}_i^0) \geqslant 0, \\
& \forall F_i^0 \in \mathscr{L}, \qquad \forall n \geqslant n^*
\end{aligned} \tag{29}$$

where $\Lambda = [(F_i^0 - \hat{F}_i^0)(F_i^0 - \hat{F}_i^0)]^{1/2}$. In this inequality the integrand on Δw_n is nonnegative a.e. and bounded, $\forall F_i^0$, $\forall n \geqslant n^*$. Moreover, the measure of Δw_n tends to zero for $n \to \infty$. As a consequence, also the value of the integral on Δw_n tends to zero for $n \to \infty$. Then the minimum condition becomes

$$\sum_j \left\{ \int_{w^+} \frac{(R_i + \hat{F}_i^0)}{\hat{T}} (F_i^0 - \hat{F}_i^0)\,ds + \Lambda \int_{w^0} (1 + \bar{\eta})\,ds \right. \\ \left. - \sum_h \left\{ \bar{X}_i^h \left[\sum_{j \in I^h} (F_i^0 - \hat{F}_i^0) - \sum_{j \in E^h} (F_i^0 - \hat{F}_i^0) \right] \right\} \geqslant 0, \qquad \forall F_i^0 \in \mathscr{L}. \tag{30}$$

because condition (29) is fulfilled in $\hat{P}$, $\forall n \geqslant n^*$.

Note that the end forces $\hat{F}_i^0$ meet the nodal equilibrium equations (21b), hence $F_i^0 \in \mathscr{L}$ if and only if

$$\sum_{j \in I^k} (F_i^0 - \hat{F}_i^0) - \sum_{j \in E^k} (F_i^0 - \hat{F}_i^0) = 0, \qquad k = 1, 2, \ldots, l. \tag{31}$$

If all the cables are in tension ($j \in V$) at the solution of the problem, the integral on w^0 is not present in inequality (30), moreover $w^+ \equiv (0, L)$ and $(1/\hat{T}) \in L_1(0, L)$. Therefore, the left-hand side coincides with the Gateaux differential of Φ_c in $\hat{P}$ with respect to the variables F_i^0, and with direction $(F_i^0 - \hat{F}_i^0)$. As a consequence, taking into account eqns (31), the minimum condition becomes

$$\sum_j \left\{ \int_0^L \frac{(R_i + \hat{F}_i^0)}{\hat{T}} (F_i^0 - \hat{F}_i^0)\,ds \right\} \\ - \sum_h \left\{ \bar{X}_i^h \left[\sum_{j \in I^h} (F_i^0 - \hat{F}_i^0) - \sum_{j \in E^h} (F_i^0 - \hat{F}_i^0) \right] \right\} = 0$$

for each $(F_i^0 - \hat{F}_i^0)$ which meets eqn (31), and it expresses the usual, variational stationary formulation of Φ_c in $\hat{P}$ with respect to variables F_i^0 on $\mathscr{L}$.

In order to remove the constraints on the differences $(F_i^0 - \hat{F}_i^0)$, the left-hand sides of eqns (31) are introduced in inequality (30) after multiplication by the $3l$ Lagrangean multipliers X_i^k. Let $(X_i^0)^j((X_i^L)^j)$ denote the prescribed scalar factor $\bar{X}_i^h$ or else the unknown one X_i^k, which pertains to index j if $j \in I^h$ or $j \in I^k$ ($j \in E^h$ or $j \in E^k$). Then inequality (30) takes the form

$$\sum_j \left\{ (F_i^0 - \hat{F}_i^0) \left[\int_{w^+} \frac{(R_i + \hat{F}_i^0)}{\hat{T}}\,ds + X_i^L - X_i^0 \right] + \Lambda \int_{w^0} (1 + \bar{\eta})\,ds \right\} \geqslant 0, \qquad \forall F_i^0$$

and letting

$$(\Delta \hat{x}_i)^j = - \int_{(w^+)^j} \frac{(R_i + \hat{F}_i^0)}{\hat{T}}\,ds, \qquad (L^0)^j = \int_{(w^0)^j} (1 + \bar{\eta})\,ds$$

it becomes

$$\sum_j \left\{ (F_i^0 - \hat{F}_i^0)(-\Delta \hat{x}_i + X_i^L - X_i^0) + \Lambda L^0 \right\} \geqslant 0, \qquad \forall F_i^0. \tag{32}$$

Inequality (32) is met for arbitrary values of F_i^0, if each term of the summation on the right-hand side is nonnegative for any $(F_i^0)^j \neq (\hat{F}_i^0)^j$, i.e. $\Lambda^j \neq 0$. If the term pertaining the jth cable is minimized with respect to F_i^0, one obtains

$$(F_i^0)^j = (\hat{F}_i^0)^j - \Lambda \frac{(X_i^L - X_i^0 - \Delta\hat{x}_i)}{[(X_k^L - X_k^0 - \Delta\hat{x}_k)(X_k^L - X_k^0 - \Delta\hat{x}_k)]^{1/2}}.$$

By using the above expression of $(F_i^0)^j$, inequality (32) becomes

$$\sum_j \Lambda\{L^0 - [(X_i^L - X_i^0 - \Delta\hat{x}_i)(X_i^L - X_i^0 - \Delta\hat{x}_i)]^{1/2}\} \geqslant 0$$

and arbitrariness of $\Lambda(\geqslant 0)$ implies for each cable

$$L^0 \geqslant [(X_i^L - X_i^0 - \Delta\hat{x}_i)(X_i^L - X_i^0 - \Delta\hat{x}_i)]^{1/2} \tag{33}$$

provided that $L^0 = 0$ $(\Delta\hat{x}_i = 0)$ if the jth cable is a.e. in tension (slackened) at the solution of the problem.

Inequality (33) can be immediately interpreted. Taking into account eqn (16), $\Delta\hat{x}_i$ coincides with the sum of the distances, in the ith direction, between the ends of the in tension parts of the cables $(s \in (w^+)^j)$. On the other hand, L^0 is the sum of the extents of the slackened parts $(s \in (w^0)^j)$, strain $\bar{\eta}$ being accounted. Note that $L^0 = 0$ and $w^+ \equiv (0, L)$ if the cable is a.e. in tension. As a consequence, $\Delta\hat{x}_i$ is the distance between the ends of the cable in the ith direction. In this case, inequality (33) implies merely

$$\Delta\hat{x}_i = X_i^L - X_i^0. \tag{34}$$

Equation (34) shows that the Lagrangean multipliers X_i^k, in their specifications X_i^0 and/or X_i^L, have the meaning of coordinates in the reference system $(0, x_i)$. Moreover, eqn (34) attaches to each in tension cable a couple of points, and the distance between these points is equal to the distance $\Delta\hat{x}_i$ between the ends of the cable, in the directions of the axes. Thus the Lagrangean multipliers X_i^k are the coordinates of the free nodes connected by the jth cable.

On the other hand, $X_i^0(X_i^L)$ assumes the same value $\bar{X}_i^h$, or else X_i^k, for all the cables whose initial (terminal) end is connected to the restrained node h, or to the free node k. Therefore, continuity of cables in the node is assured. Moreover, X_i^0 and X_i^L, for any in tension cable, are univocally defined if at least one end of the cable is connected to a restrained node, directly or by a sequence of in tension cables. Otherwise, X_i^0, X_i^L are determined apart from an arbitrary constant (the same for X_i^0, X_i^L), corresponding to a rigid body translation in the direction of the axes, which leaves the traction along the cable unaffected.

If slackened parts are present on a cable $(L^0 > 0)$, inequality (33) shows that the difference between the vector of components $(X_i^L - X_i^0)$ and the one of components $\Delta\hat{x}_i$, is a vector of modulus less than L^0, which is the sum of the lengths of the slackened parts. In this case, inequality (33) cannot, by itself, define the coordinates of one end of the cable, once the coordinates of the other end are fixed. In other words, the position of each end of the cable can be defined only if this end is connected to a restrained node, directly or through a sequence of in tension cables. Finally, for a (a.e.) slackened cable $(\Delta\hat{x}_i = 0)$, inequality (33) takes the form

$$L^0 \geqslant [(X_i^L - X_i^0)(X_i^L - X_i^0)]^{1/2}$$

which means, as natural, that the length of an unstressed cable cannot be less than the distance between the nodes connected by it.

As a consequence, eqn (33) expresses the boundary compatibility conditions for the cable in the present context.

Conditions (26) and (33) have been proved to be necessary for fulfilling inequality (24), but they are also sufficient. Indeed, let condition (33) be true for each cable. Then inequality (32) is fulfilled for each F_i^0, and inequality (30) is fulfilled too, as inequality (32) coincides with inequality (30) for each $F_i^0 \in \mathscr{L}$. Thus, also inequality (29) is fulfilled, as the integral on Δw_n is nonnegative for each n. This inequality is obtained by minimizing the left-hand side of inequality (27) on the set of the admissible T's, for any given $F_i^0 \in \mathscr{L}$. As a consequence, if inequality (29) is true for any $F_i^0 \in \mathscr{L}$, also inequality (27) is true for each $P \in \mathscr{D}$, and inequality (24) (remember condition (26)) is fulfilled. Hence conditions (26) and (33) are sufficient for the minimum of Φ_c on $\mathscr{D}$.

5. APPLICATIONS AND CONCLUDING REMARKS

The exposed theory is applied to the solution of some sample problems. A single cable is considered, strains are assumed negligible. The actual traction $T^* = T(1 + \eta)$ is related to the relative elongation $\eta_{el} = \eta - \bar{\eta}$ through the relationship

$$T^* = EA\eta_{el} = EA(\eta - \bar{\eta})$$

E being the elastic modulus of the cable, A its cross-sectional area. The strain energy density function reads

$$G(T) = \frac{1}{2}(1 + 2\bar{\varepsilon})\frac{T^2}{EA - T}, \qquad \text{with} \quad \bar{\varepsilon} = \bar{\eta}\left(1 + \frac{\bar{\eta}}{2}\right).$$

Lagrangean traction T is represented, according to the usual finite element technique, as a piecewise constant (one stress parameter per element) or linear (two stress parameters per element) function of the abscissa s. The *a priori* transitional equilibrium conditions at the interelement should be imposed on the actual traction T^* for obtaining a really stress-diffusive (equilibrium) model. Therefore, componentwise continuity on terminal forces F_i must be kept in the assembling process between consecutive elements at their common node, if no lumped load acts on the node. More generally, equilibrium among the terminal forces and the acting load should be enforced *a priori* at a node. From a geometrical standpoint, this condition implies continuity of the tangent to the cable at a node for any possible configuration C, if the external load on the node is zero.

The loading schemes considered allow one to anticipate that the cable will be all in tension at the equilibrium configuration. As a consequence, functional (17) is differentiable at the solution. After discretization of the interval $0 \leqslant s \leqslant L$, and algebrization of the functional, this one becomes a non-linear, convex function of the stress parameters, and it can be differentiated at its minimum point. The gradient of this function is set equal to zero, and a system of non-linear equations is obtained. This system is solved for stress parameters, and evaluation of tractions as well as displacements, by integrating eqn (16), is then possible.

Figure 1 depicts a problem considered by Ozdemir[15] and Jayaraman and Knudson[16].

The unstrained length L_A of the cable in this problem is less than the distance between the supports, hence the cable is pre-tensioned. An exact solution is available[13]. Tables 1 and 2 collect the values of restraint reaction F_x^0 (i.e. the horizontal component of traction T^*), and deflection f at the midspan. In the author's approach, the cable has been subdivided into elements of the same length, with T, respectively, constant (constant stress element, CSE), and linear (LSE). NV is the number of stress parameters for the whole assembly, in addition to the two reaction components at the initial end of the cable. Note that Knudson and Odzemir give no result for tractions.

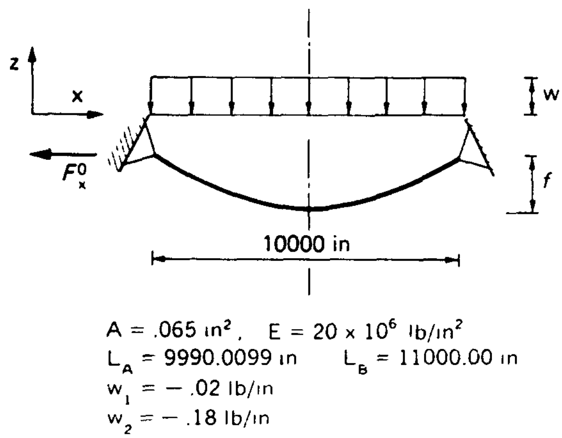

Fig. 1.

Table 1. Distributed load $w_1 = -0.02\,\mathrm{lb\,in^{-1}}$, $L_A = 9990.0099$ in

NV	F_x^0 (CSE) lb	F_x^0 (LSE) lb	f (CSE) in	f (LSE) in
1	−1899.13		−131.518	
2	−1899.13	−1899.13	−131.518	−131.518
4	−1898.99	−1898.94	−131.495	−131.490
6	−1898.96	−1898.93	−131.493	−131.492
8	−1898.94	−1898.93	−131.492	−131.491
16	−1898.93	−1898.93	−131.492	−131.491
Exact solution	−1898.93		−131.491	
Knudson	—		−131.63	
Ozdemir	—		−131.60	

Table 2. Distributed load $w_2 = -0.18\,\mathrm{lb\,in^{-1}}$, $L_A = 9990.0099$ in

NV	F_x^0 (CSE) lb	F_x^0 (LSE) lb	f (CSE) in	f (LSE) in
1	−6053.48		−371.315	
2	−6053.48	−6053.48	−371.315	−371.315
4	−6047.78	−6045.90	−371.158	−371.105
6	−6046.49	−6045.49	−371.144	−371.139
8	−6045.02	−6045.43	−371.140	−371.134
10	−6045.79	−6045.41	−371.139	−371.137
12	−6045.67	−6045.40	−371.138	−371.136
24	−6045.47	−6045.40	−371.137	−371.136
Exact solution	−6045.40		−371.136	
Knudson	—		−371.13	
Ozdemir	—		−368.00	

As the problem is symmetric, the descriptions accomplished with one or two elements CSE (NV = 1, 2), as well as one element LSE (NV = 2), for the whole cable, lead to the same results. Note the results obtained with the present approach with one element CSE (1 + 2 variables) only, or else with two elements LSE (4 + 2 variables). It is worth observing that an increase in the intensity of the load causes less accurate results, the number of elements being the same. This fact can be explained by noting that the higher the load, the higher the traction change along the cable.

The case of the same cable, but with unstrained length L_B larger than the distance between the supports, has also been analyzed, see Table 3. Convergence seems slower than in the previous example, but by no means less satisfactory.

In spite of their simplicity, the examples exposed lead one to believe that quite good results are achievable by the proposed approach with a very moderate effort.

Table 3. Distributed load $w_2 = -0.18\,\mathrm{lb\,in^{-1}}$, $L_B = 11000.0\,\mathrm{in}$

NV	F_x^0(CSE) lb	F_x^0 (LSE) lb	f (CSE) in	f (LSE) in
1	−1239.80		−1996.29	
2	−1239.80	−1239.80	−1996.29	−1996.29
4	−1189.87	−1174.74	−2006.64	−2008.21
6	−1179.89	−1172.29	−2010.75	−2015.06
8	−1176.29	−1172.00	−2012.41	−2014.29
10	−1174.60	−1171.67	−2013.22	−2014.77
12	−1173.68	−1171.62	−2013.67	−2014.64
14	−1173.12	−1171.60	−2013.95	−2014.74
16	−1172.76	−1171.59	−2014.13	−2014.70
18	−1172.51	−1171.58	−2014.25	−2014.73
20	−1172.33	−1171.57	−2014.34	−2014.71
30	−1171.91	−1171.57	−2014.55	−2014.72
Exact solution	−1171.57		−2014.72	

Finally, it should be remarked that in passing from a plane problem to a three-dimensional one with the same discretization, only one variable should be added for the whole cable: the third component of reaction at the initial end of the cable. Such a peculiarity could lead, by itself, one to consider favourably the proposed approach.

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REFERENCES

1. C. H. Thornton and C. Birnstiel, Three-dimensional suspension structures. *J. Struct. Div. A.S.C.E.* **93**, 247–270 (1967).
2. S. A. Saafan, Theoretical analysis of suspension roofs. *J. Struct. Div. A.S.C.E.* **96**, 392–405 (1970).
3. B. R. Buchholdt and B. R. McMillan, Iterative methods for the solution of pretensioned cable structures and pinjointed assemblies having significant geometrical displacements. *Proc. 1971 IASS Pacific Symposium Part II on Tension Structures and Space Frames*, Tokyo and Kyoto, pp. 305–316 (1971).
4. J. H. Argyris and D. W. Scharpf, Large deformation analysis of prestressed networks. *J. Struct. Div. A.S.C.E.* **48**, 633–654 (1972).
5. D. P. Greenberg, Inelastic analysis of suspension roof-structures. *J. Struct. Div. A.S.C.E.* **96**, 905–930 (1970).
6. J. J. Jonatowsky and C. Birnstiel, Inelastic stiffened suspension space structures. *J. Struct. Div. A.S.C.E.* **96**, 1143–1166 (1970).
7. R. Contro, G. Maier and A. Zavelani, Inelastic analysis of suspension structures by nonlinear programming. *Comp. Meth. Appl. Mech. Engng* **5**, 127–143 (1975).
8. W. T. O'Brien, General solution of suspended cable problems. *J. Struct. Div. A.S.C.E.* **93**, 1–26 (1967).
9. H. H. West and A. R. Robinson, Continuous method of suspension bridge analysis. *J. Struct. Div. Proc. A.S.C.E.* **94**, 2861–2883 (1968).
10. A. H. Peyrot and A. M. Goulois, Analysis of flexible transmission lines. *J. Struct. Div. A.S.C.E.* **104**, 763–779 (1978).
11. A. H. Peyrot and A. M. Goulois, Analysis of cable structures. *Computers Structures* **10**, 805–813 (1979).
12. J. P. Lemaire, Etude de systèmes de cables. *Annls Inst. Tech. Bâtim.* No. 361, 111–134 (May 1978).
13. G. B. Sinclair and S. B. Hodder, Exact solution for elastic cable systems. *Int. J. Solids Structures* **17**, 845–854 (1981).
14. W. H. Hengold and J. J. Russell, Equilibrium and natural frequencies of cable structures (a nonlinear finite element approach). *Computers Structures* **6**, 267–271 (1976).
15. H. Ozdemir, A finite element approach for cable problems. *Int. J. Solids Structures* **15**, 427–437 (1979).
16. H. B. Jayaraman and W. C. Knudson, A curved element for the analysis of cable structures. *Computers Structures* **14**, 325–333 (1981).
17. P. D. Panagiotopoulos, A variational inequality approach to the inelastic stress-unilateral analysis of cable structures. *Computers Structures* **6**, 133–139 (1976).
18. M. Cannarozzi, Stationary and extremum variational formulations for the elastostatics of cable networks. *Meccanica* **20**, 136–143 (1985).
19. H. L. Royden, *Real Analysis*, p. 128. Macmillan, New York (1968).
20. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, p. 11. North-Holland, Amsterdam (1976).
21. K. Yosida, *Functional Analysis*, p. 140. Springer, Berlin (1978).
22. M. M. Vainberg, *Variational Method and Method of Monotone Operators*, p. 93. Wiley, New York (1973).
23. R. T. Rockafellar, *Convex Analysis*, p. 83. Princeton University Press, Princeton, New Jersey (1972).
24. W. Ledermann, *Handbook of Applicable Mathematics*, Vol. IV, p. 169. Wiley, New York (1982).

APPENDIX I

Let

$$\Phi_c(T, F_i^0) = \Phi_1 + \Phi_2 + \Phi_3 + \mathscr{F}$$

where

$$\Phi_1 = \sum_j \int_0^L G(T)\,\mathrm{d}s, \qquad \Phi_2 = \sum_j \int_0^L \frac{1}{2T}(R_i + F_i^0)(R_i + F_i^0)\,\mathrm{d}s$$

$$\Phi_3 = \sum_j \int_0^L T\left(\frac{1}{2} + \bar{\varepsilon}\right)\mathrm{d}s$$

$$\mathscr{F} = -\sum_h \left\{ X_i^h \left[\sum_{j \in I^h} (F_i^0) - \sum_{j \in E^h} (R_i(L) + F_i^0) \right] \right\}.$$

Then functional Φ_c is

(a) strictly convex,
(b) coercive,
(c) lower semicontinuous (l.s.c.) on $\mathscr{D} \equiv \mathscr{W} \cap \mathscr{L}$, $\mathscr{D} \subset \mathscr{S}$.

(a) $\mathscr{L}$ is a linear manifold in $\mathscr{S}$, hence convexity of $\Phi_c(T, F_i^0)$ on $\mathscr{W}$ implies convexity on $\mathscr{D}$ for Φ_c.

Note that Φ_1 is strictly convex with regard to T, as G is strictly convex. Moreover, Φ_3 and $\mathscr{F}$ are weakly convex (linear) with respect to T and F_i^0, respectively.

Consider now the points $\hat{P} \equiv (\hat{T}, \hat{F}_i^0)$ and $\bar{P} \equiv (\bar{T}, \bar{F}_i^0)$, $\hat{P}, \bar{P} \in \mathscr{W}$. Then Φ_c is convex on $\mathscr{W}$ because inequality

$$\Phi_2(\lambda\hat{P} + (1-\lambda)\bar{P}) \leqslant \lambda\Phi_2(\hat{P}) + (1-\lambda)\Phi_2(\bar{P}), \qquad \lambda \in (0,1) \tag{A1}$$

is fulfilled. Indeed, the integrand in Φ_2 (see defs (17c), (17e) and (17f) of function φ_2) is a convex function[23].

As Φ_1 is strictly convex with respect to T, $\Phi_c(T, F_i^0)$ is strictly convex if inequality (A1) is strictly fulfilled for $\hat{T} = \bar{T}$. Assume $\hat{T} = \bar{T} = T \geqslant 0$, then inequality (A1) becomes

$$\lambda(1-\lambda)\Sigma_j \int_0^L \frac{[\hat{F}_i^0 - \bar{F}_i^0][\hat{F}_i^0 - \bar{F}_i^0]}{T}\,\mathrm{d}s \geqslant 0, \qquad \lambda \in (0,1)$$

and is strictly fulfilled if $\hat{F}_i^0 \neq \bar{F}_i^0$, $T \geqslant 0$.

(b) Let u be an element of the set $\mathscr{D} \subset \mathscr{S}$, then $\Phi_c(T, F_i^0)$ is coercive if

$$\lim_{\|u\| \to \infty} \Phi_c(u) = \infty$$

and this is true if

$$\lim_{\|u\| \to \infty} \frac{\Phi_c(u)}{\|u\|} = \infty. \tag{A2}$$

As Φ_3 and $\mathscr{F}$ are respectively a linear functional and a linear function, condition (A2) is fulfilled if

$$\lim_{\|u\| \to \infty} \frac{\Phi_1(u) + \Phi_2(u)}{\|u\|} = \infty. \tag{A3}$$

Let Φ_1^j and Φ_2^j be the jth contributions to Φ_1 and Φ_2. Φ_1^j and Φ_2^j depend on vector $v \equiv \{T^j \mid (F_i^0)^j\}$, $v \in L_2(0, L^j) \times \mathbf{R}^3$, $T^j \geqslant 0$, $\|v\| = \{\|T^j\|^2 + \|(F_i^0)^j\|^2\}^{1/2}$, $\Phi_1^j \geqslant 0$ and $\Phi_2^j \geqslant 0$, $\forall v$. It can be easily proved that condition (A3) is fulfilled if

$$\lim_{\|v\| \to \infty} \frac{\Phi_1^j(v) + \Phi_2^j(v)}{\|v\|} = \infty, \qquad \forall j. \tag{A4}$$

Note that $\mathrm{d}\varepsilon/\mathrm{d}T = \mathrm{d}^2G/\mathrm{d}T^2 \geqslant \alpha > 0$ (eqn (7d)), hence

$$\Phi_1^j = \int_0^{L^j} G(T)\,\mathrm{d}s \geqslant \frac{\alpha}{2}\int_0^{L^j} T^2\,\mathrm{d}s = \frac{\alpha}{2}\|T\|^2.$$

Consider the sequence $\{v_n\} = \{T_n, (F_i^0)_n\}: \lim_n \|v_n\| = \infty$. As $\|v_n\| \leqslant \|T_n\| + \|(F_i^0)_n\|$, it results $\lim_n \|T_n\| = \infty$ and/or $\lim_n \|(F_i^0)_n\| = \infty$.

The subsequent cases are possible:

(i) $\lim_n \|v_n\| = \infty$ and $\lim_n \frac{\|(F_i^0)_n\|}{\|T_n\|} = H < \infty$, as a consequence $\lim_n \|T_n\| = \infty$.

(ii) $\lim_n \|v_n\| = \infty$ and $\lim_n \frac{\|(F_i^0)_n\|}{\|T_n\|} = \infty$, as a consequence $\lim_n \|(F_i^0)_n\| = \infty$.

Case (i). The continuous inequality holds

$$\frac{\Phi_1^j + \Phi_2^j}{\|v_n\|} \geqslant \frac{\Phi_1^j}{\|v_n\|} \geqslant \frac{\alpha}{2}\frac{\|T_n\|^2}{\|v_n\|} \geqslant \frac{\alpha}{2}\frac{\|T_n\|^2}{\|T_n\| + \|(F_i^0)_n\|} = \frac{\alpha}{2}\frac{\|T_n\|}{1 + \frac{\|(F_i^0)_n\|}{\|T_n\|}} = \psi_n.$$

Because of $\lim_n \psi_n = \infty$, condition (A4) is met.

Case (ii). Consider the sequence $\{T_n\}$. For the mean value theorem, $\exists\, \alpha_n$, $\inf(T_n(s)) \leqslant \alpha_n \leqslant \sup(T_n(s))$, $s \in (0, L)$, such that

$$\alpha_n L = \int_0^L T_n \,\mathrm{d}s.$$

For a fixed $\Delta > 1$, $\beta_n = \Delta\alpha_n$, let the sets $w_n = \{s \mid T_n(s) \leqslant \beta_n\}$ (of positive measure), and $z_n = \{s \mid T_n(s) > \beta_n\}$ (of possibly zero measure). The measure of $(\cdot)$ is denoted by $\mu(\cdot)$. Inequality

$$\beta_n \mu(z_n) < \int_{z_n} T_n \,\mathrm{d}s \leqslant \int_0^L T_n \,\mathrm{d}s = \alpha_n L$$

holds. As $\mu(w_n) + \mu(z_n) = L$, it is also $\beta_n(L - \mu(w_n)) < \alpha_n L$, and consequently $\mu(w_n) > L(1 - 1/\Delta)$, $\forall n$. Observe that

$$L^{-1}\|T_n\|_1 = L^{-1}\int_0^L |T_n| \,\mathrm{d}s \leqslant L^{-1/2}\|T_n\|$$

hence

$$\int_0^L T_n \,\mathrm{d}s \leqslant L^{1/2}\|T_n\|$$

because of $T_n \geqslant 0$, and finally

$$\alpha_n \leqslant L^{-1/2}\|T_n\| \quad \text{and} \quad \beta_n \leqslant \Delta L^{-1/2}\|T_n\|.$$

As a consequence, the following inequality holds

$$\begin{aligned}
\frac{\Phi_1^j + \Phi_2^j}{\|v_n\|} &\geqslant \frac{\Phi_2^j}{\|v_n\|} = \frac{1}{\|v_n\|}\int_0^L \frac{[R_i + (F_i^0)_n][R_i + (F_i^0)_n]}{T_n}\mathrm{d}s \\
&\geqslant \frac{1}{\|T_n\| + \|(F_i^0)\|}\frac{1}{\beta_n}\int_{w_n} [R_i + (F_i^0)_n][R_i + (F_i^0)_n]\,\mathrm{d}s \\
&\geqslant \frac{L^{1/2}}{\Delta\|T_n\|\{\|T_n\| + \|(F_i^0)_n\|\}}\int_{w_n} [R_i + (F_i^0)_n][R_i + (F_i^0)_n]\,\mathrm{d}s \\
&= \frac{L^{1/2}}{\Delta}\frac{\|(F_i^0)_n\|^2}{\|T_n\|\{\|T_n\| + \|(F_i^0)_n\|\}}\int_{w_n}\left[\frac{R_i + (F_i^0)_n}{\|(F_i^0)_n\|}\right]\left[\frac{R_i + (F_i^0)_n}{\|(F_i^0)_n\|}\right]\mathrm{d}s \\
&= \frac{L^{1/2}}{\Delta}\frac{\|(F_i^0)_n\|}{\|T_n\|}\frac{1}{\frac{\|T_n\|}{\|(F_i^0)_n\|} + 1}\int_{w_n}\left[\frac{R_i + (F_i^0)_n}{\|(F_i^0)_n\|}\right]\left[\frac{R_i + (F_i^0)_n}{\|(F_i^0)_n\|}\right]\mathrm{d}s = \psi_n.
\end{aligned}$$

Observe that the functions $R_i(s)$ are bounded on $(0, L)$, and $\lim_n \|(F_i^0)_n\| = \infty$, hence

$$\lim_n \int_{w_n} (\cdot)\,\mathrm{d}s = \mu(w_n) > L(1 - 1/\Delta).$$

As $\lim_n \frac{\|(F_i^0)_n\|}{\|T_n\|} = \infty$ implies $\lim_n \frac{\|T_n\|}{\|(F_i^0)_n\|} = 0$, $\lim_n \psi_n = \infty$ follows, and condition (A4) is met.

(c) Observe that Φ_3 and $\mathscr{F}$ are linear and continuous with respect to T and F_i^0, respectively. By this fact, Φ_r is l.s.c. if also $\psi = \Phi_1 + \Phi_2$ is so. Moreover, the integrands in Φ_1 and Φ_2 take non-negative values for each $(T, F_i^0) \in \mathscr{D}$. Let, for a fixed real number k, be the set

$$w_k = \{(T, F_i^0) \mid \psi(T, F_i^0) \leqslant k, (T, F_i^0) \in \mathscr{D}\}$$

where ψ is l.s.c. if w_k is strongly closed[20, pp. 9, 10], i.e. if any convergent sequence $\{T_n, (F_i^0)_n\}$ in w_k converges *in the norm of* $\mathscr{S}$ *to an element of* w_k.

Consider a sequence $\{T_n, (F_i^0)_n\}$ and let $\lim_n \{T_n, (F_i^0)_n\} = (T, F_i^0)$.

Then, there exists a subsequence $\{T_{n_k}, (F_i^0)_{n_k}\}$ of the above sequence, which converges to (T, F_i^0) a.e., and $\psi[T_{n_k}, (F_i^0)_{n_k}] \leqslant k$, because this subsequence belongs to w_k. On the other hand, Fatou's lemma[24] gives

$$\psi[\lim \{T_{n_k}, (F_i^0)_{n_k}\}] = \psi[T, F_i^0] \leqslant \liminf \psi[\{T_{n_k}, (F_i^0)_{n_k}\}] \leqslant k$$

as a consequence $(T, F_i^0) \in w_k$, and ψ is l.s.c.

APPENDIX II

Let $\delta\psi(\hat{P}; t, f_i)$ the Gateaux differential of the functional $\psi(T, F_i^0)$, $(T, F_i^0) \in L_2(w) \times \mathbb{R}^3$ *in the point* $\hat{P} \equiv (\hat{T}, \hat{F}_i^0)$, with direction (t, f_i). The direction (t, f_i) is admissible if $\psi(\hat{T} + \lambda t, \hat{F}_i^0 + \lambda f_i)$ is defined for sufficiently small values of $\lambda \geqslant 0$. The Gateaux differential in the point

$$\hat{P} \equiv \{(\hat{T}, \hat{F}_i^0) \mid \hat{T} > 0 \text{ a.e.}, \varphi_2(\hat{T}, \hat{F}_i^0) \in L_1(w)\}$$

is

$$\begin{aligned}\delta\psi(\hat{P}; t, f_i) &= \lim_{\lambda \to 0^+} \int_w \left\{ -\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{2\hat{T}(\hat{T} + \lambda t)} t + \lambda \frac{f_i f_i}{2(\hat{T} + \lambda t)} + \frac{(R_i + \hat{F}_i^0)}{(\hat{T} + \lambda t)} f_i \right\} \mathrm{d}s \\ &= \lim_{\lambda \to 0^+} \int_w \omega(\lambda) \, \mathrm{d}s. \end{aligned} \tag{A5}$$

As (t, f_i) is an admissible direction, there exists a $\bar{\lambda}$ such that $(\hat{T} + \lambda t) \geqslant 0$, $\forall \lambda: 0 \leqslant \lambda \leqslant \bar{\lambda}$, hence the inequality $(\hat{T} + \lambda t) \geqslant \hat{T}/2$ holds for any λ such that $0 \leqslant \lambda \leqslant \bar{\lambda}/2$. Let $\lambda^* = \min\{1, \bar{\lambda}/2\}$, then the following inequalities:

$$\begin{aligned}|\omega(\lambda)| &\leqslant \left| -\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{2\hat{T}(\hat{T} + \lambda t)} t \right| + \left| \lambda \frac{f_i f_i}{2(\hat{T} + \lambda t)} \right| + \left| \frac{(R_i + \hat{F}_i^0)}{(\hat{T} + \lambda t)} f_i \right| \\ &\leqslant \left| -\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}^2} t \right| + \left| \frac{f_i f_i}{\hat{T}} \right| + \left| 2\frac{(R_i + \hat{F}_i^0)}{\hat{T}} f_i \right| \\ &= \left| 2\left[\frac{\partial \varphi_2}{\partial T}\right]_{\hat{P}} t \right| + \left| \frac{1}{\hat{T}} f_i f_i \right| + \left| 2\left[\frac{\partial \varphi_2}{\partial F_i^0}\right]_{\hat{P}} f_i \right| = |\chi_1| + |\chi_2| + |\chi_3|\end{aligned}$$

hold for any $\lambda \in (0, \lambda^*)$.

Observe that $[\partial\varphi_2/\partial T]_{\hat{P}} \in L_2(w)$ implies $[\partial\varphi_2/\partial F_i^0]_{\hat{P}} \in L_1(w)$. Indeed

$$\left\{ \left[\frac{\partial \varphi_2}{\partial F_i^0}\right]_{\hat{P}} \left[\frac{\partial \varphi_2}{\partial F_i^0}\right]_{\hat{P}} \right\}^2 = 4 \left[\frac{\partial \varphi_2}{\partial T}\right]_{\hat{P}} \left[\frac{\partial \varphi_2}{\partial T}\right]_{\hat{P}} \in L_1(w)$$

moreover

$$\left| \left[\frac{\partial \varphi_2}{\partial F_k^0}\right]_{\hat{P}} \right| \leqslant 1 + \left\{ \left[\frac{\partial \varphi_2}{\partial F_i^0}\right]_{\hat{P}} \left[\frac{\partial \varphi_2}{\partial F_i^0}\right]_{\hat{P}} \right\}^2, \qquad k = 1, 2, 3$$

and consequently $[\partial\varphi_2/\partial F_i^0]_{\hat{P}} \in L_1(w)$.

Observe that the functions $|\chi_1|, |\chi_2|$ and $|\chi_3|$ belong to $L_1(w)$ if conditions

$$(1) \quad \left[\frac{\partial \varphi_2}{\partial T}\right]_{\hat{P}} \in L_2(w), \qquad (2) \quad \frac{1}{\hat{T}} \in L_1(w)$$

hold, as $t \in L_2(w)$. Then $|\omega(\lambda)|$ is dominated by a function belonging to L_1 for each $\lambda \in (0, \lambda^*)$. Hence the limit exists and takes the form

$$\delta\psi(\hat{P}; t, f_i) = \left\langle \left[\frac{\partial\varphi_2}{\partial T}\right]_{\hat{P}}, t \right\rangle + f_i \int_w \left[\frac{\partial\varphi_2}{\partial F_i^0}\right]_{\hat{P}} \mathrm{d}s$$

and $\psi(T, F_i^0)$ is Gateaux differentiable in $\hat{P}$.

If condition (2) is not fulfilled, there exists only $\delta\psi(\hat{P}; t, 0)$, and ψ is differentiable only with respect to T.

APPENDIX III

Equation (26) implies that $\mathrm{d}G/\mathrm{d}T|_{\hat{T}}$ is bounded on $(w_{n^*}^+)$ for $n = n^*$, because the functions R_i and $\bar{\varepsilon}$ are bounded on $(0, L)$, $\hat{T}(s) > 1/n^*$, $s \in (w_{n^*}^+)$. Let $\alpha = \sup(\mathrm{d}G/\mathrm{d}T|_{\hat{T}})$ on $(w_{n^*}^+)$, let $\beta = \sup(\bar{\varepsilon})$ on $(0, L)$, and observe that $\mathrm{d}G/\mathrm{d}T$ is a strictly increasing function of T. Then it follows that $\mathrm{d}G/\mathrm{d}T|_{\hat{T}} \leqslant \alpha$ on (w_n^+), $\forall n > n^*$, and eqn (26) implies

$$\frac{1}{2}\frac{(R_i + \hat{F}_i^0)(R_i + \hat{F}_i^0)}{\hat{T}^2} = -\left[\frac{\partial\varphi_2}{\partial T}\right]_{\hat{T}, \hat{F}_i^0} = \left.\frac{\mathrm{d}G}{\mathrm{d}\hat{T}}\right|_{\hat{T}} + \frac{1}{2} + \bar{\varepsilon} \leqslant \alpha + \beta + \frac{1}{2}$$

on (w_n^+), $\forall n > n^*$. As the function $(-[\partial\varphi_2/\partial T]_{\hat{T}, \hat{F}_i^0})$ is nonnegative and bounded from above independently of n, this function belongs to $L_\infty(w^+)$.